

# The simple method of distinguishing the underlying differentiable structures of algebraic surfaces

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## 0 Introduction

The purpose of this preprint is to construct a new invariant of the smooth structure of a simply connected 4-manifold  $M$  so called Spin-polynomials

$$\gamma_1^{g,C}(2, c_1, c_2) \in S^{d_1} H^2(M, \mathbb{Z}) \quad (0.1)$$

and to show how to use it to compare the smooth structures of rational surfaces and surfaces of general type.

This Spin-polynomial (0.1) is the analogue of the original Donaldson polynomial

$$\gamma^g(2, c_1, c_2) \in S^d H^2(M, \mathbb{Z}) \quad (0.1')$$

and depends on one extra index  $C$  given by a so called  $Spin^{\mathbb{C}}$ -structure on  $M$ . It specifies a lift of the Stiefel-Whitney class  $w_2(M) \in H^2(M, \mathbb{Z}_2)$  to some integer class  $C \in H^2(M, \mathbb{Z})$ .

The technical basis of the construction of the Spin-polynomial is the same as of the ordinary polynomials (0.1'). It is our aim to compare the properties of polynomials (0.1) and (0.1')

Here our aim is to consider only the simplest version of invariants of such type and to use in applications the simplest arguments of proofs (now standart for Donaldson's stuff ).

Much more sophisticated constructions and a vague discussion of the properties of these invariants are contained in forthcoming article [T 3].

A special case of our construction has been used in the article [P-T], where some important basic theorems were proved. By this reason we will follow the english translation of this article in terminology and notation (see [P-T]).

As applications of our techniques we will prove the non existence of algebraic fake planes, Hirzebruch surfaces or quadrics. (Here "fake" means "diffeomorphic to ..., but algebraically non equivalent").

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## 1 $Spin^{\mathbb{C}}$ -structure

The Stiefel-Whitney class  $w_2(M) \in H^2(M, \mathbb{Z}_2)$  of a smooth, compact 4-manifold M is the characteristic class of the lattice  $H^2(M, \mathbb{Z})$  with its intersection form  $q_M$ . This means that for every  $\sigma \in H^2(M, \mathbb{Z})$

$$\sigma^2 \equiv \sigma \cdot w_2(M) \mod 2 \quad (1.1)$$

**Definition 1.1.** Let M be a smooth, simply connected, compact 4-manifold, then a class  $C \in H^2(M, \mathbb{Z})$  such that  $C \equiv w_2(M) \mod 2$  is called a  $Spin^{\mathbb{C}}$ -structure of M. Thus to equip M with a  $Spin^{\mathbb{C}}$ -structure is the same as to lift up  $w_2$  to an integer class.

The set of all  $Spin^{\mathbb{C}}$ -structures on M is the affine sublattice

$$H_w(M) = \{\sigma \in H^2(M, \mathbb{Z}) \mid \sigma \equiv w_2(M) \mod 2\} \quad (1.2)$$

**Remarks** 1) For every  $C \in H_w(M)$

$$C^2 \equiv b_2^+ - b_2^- = I \mod 8 \quad (1.3)$$

(here I is the index or signature of M)

2) The diffeomorphism group  $\text{Diff } M$  of M acts on  $H_w(M)$  by the affine transformations.

3) the lattice  $H^2(M, \mathbb{Z})$  1-transitive acts on  $H_w(M)$  by the formula: for  $C \in H_w(M)$

$$\sigma(C) = C + 2\sigma \quad (1.4)$$

Hence, the choice of a  $C_0$  of  $H_w(M)$  gives an identification :  $H_w(M) = H^2(M)$ .

For the rest of this paper the symbol  $L_\sigma$  for  $\sigma \in H^2(M, \mathbb{Z})$  will denote a complex line bundle with the first Chern class

$$c_1(L_\sigma) = \sigma \quad (1.5)$$

If  $M$  is equipped with a Riemannian metric  $g$  the  $Spin^{\mathbb{C}}$ -structure  $C$  on  $M$  defines a pair of rank 2 Hermitian vector bundles  $W^+$  and  $W^-$  such that the complexification of the tangent bundle can be decomposed as a tensor product

$$TM_{\mathbb{C}} = (W^-)^* \otimes W^+ \quad (1.6)$$

with

$$\Lambda^2 W^{+, -} = L_C \quad (1.7)$$

Moreover, if we equip  $L_C$  with any Hermitian connection  $\nabla_0$ , then for any  $U(2)$ -bundle  $E$  on  $M$  and any Hermitian connection

$$a \in A_h(E) \quad (1.8)$$

on  $E$  we have a coupled Dirac operator

$$D_a^{C, \nabla_0} : \Gamma^\infty(E \otimes W^+) \rightarrow \Gamma^\infty(E \otimes W^-) \quad (1.9)$$

and an integer number

$$\chi_C(E) = \text{ind} D_a^{C, \nabla_0} \quad (1.10)$$

which is the index of the Fredholm operator (1.9).

It is easy to see that this integer number doesn't depend on the continuous parameters  $g$ ,  $\nabla_0$  and  $a$  and depends on the  $Spin^{\mathbb{C}}$ -structure  $C \in H_w(M)$  only.

If we change the  $Spin^{\mathbb{C}}$ -structure from  $C$  to  $C'$  (we can consider it as a result of the action (1.4) of  $\delta = (C' - C)/2$  on  $C$ ) we have

$$\chi_{C+2\delta}(E) = \chi_C(E) + c_1 \cdot \delta - \delta(\delta + C) \quad (1.11)$$

because of

$$\chi_{C+2\delta}(E) = \chi_C(E \otimes L_\delta) \quad (1.12)$$

and by the Atiyah - Singer formula

$$\chi_C(E) = c_1(c_1 + C)/2 + 2\chi_C(L_o) - c_2 \quad (1.13)$$

.Here  $L_o$  denotes the trivial line bundle on  $M$  and  $c_1, c_2$  are the Chern- classes of  $E$ .

Finally we should point out that special structures on  $M$  can define a canonical  $Spin^{\mathbb{C}}$ -structure: for example if  $M$  is the underlying structure of a complex surface  $S$ , then there is the natural  $Spin^{\mathbb{C}}$ -structure  $C = c_1(S) = -K_S$  given by the anticanonical class or if  $M$  admits a symplectic structure  $\omega$ , then there exists the canonical set of complex structures on the tangent bundle  $TM=E$  with the same first Chern class  $c_1(M, \omega)$  which is the natural  $Spin^{\mathbb{C}}$ -structure for  $(M, \omega)$ .

## 2 The definition of Spin-polynomials

If a  $Spin^{\mathbb{C}}$ -4-manifold  $(M, C)$  is equipped with a Riemannian metric  $g$  then for every  $U(2)$ - bundle  $E$  the gauge - orbit space

$$\mathcal{B}(E) = \mathcal{A}_h^*(E)/\mathcal{G}$$

of irreducible connections contains the subspace

$$\mathcal{M}^g(E) \subset \mathcal{B}(E) \quad (2.1)$$

of anti self dual connections with respect to the Riemannian metric  $g$ .

We can consider the subspace of  $\mathcal{M}^g(E)$ :

$$\mathcal{M}_1^{g,C}(E) = \{(a) \in \mathcal{M}^g(E) | rk \ ker D_a^{C, \nabla^0} \geq 1\} \quad (2.2)$$

Analogously,

$$\mathcal{M}_2^{g,C}(E) = \{(a) \in \mathcal{M}^g(E) | rk \ ker D_a^{C, \nabla^0} \geq 2\}$$

and so on.

If  $(a) \in \mathcal{M}_1^{g,C} - \mathcal{M}_2^{g,C}$  and the family of Dirac operators is in "general position" near  $(a)$ , then the fibre of the normal bundle to  $\mathcal{M}_1^{g,C}(E)$  at  $(a)$  is given by

$$(N_{\mathcal{M}_1 \subset \mathcal{M}})_{(a)} = \text{Hom}(\ker D_a, \text{coker } D_a) \quad (2.3)$$

with  $\ker D_a = \mathbb{C}$ ,  $\text{coker } D_a = \mathbb{C}^{1-\chi_C(E)}$  (if the index of the Dirac operators is not positive ).

Thus the virtual (expected) codimension of  $\mathcal{M}_1^{g,C}(E)$

$$v.\text{codim} \mathcal{M}_1^{g,C}(E) = 2 - 2\chi_C(E) \quad (2.4)$$

On the analogy of the Freed - Uhlenbeck theorem, which says that for generic metric  $g$  the moduli space  $\mathcal{M}^g(E)$  (2.1) is a smooth manifold of the expected dimension with regular ends (see Theorem 3.13 of [F-U]) the following fact was proved in section 3 of Ch.2 of [P-T] .

**Transversality Theorem.** For generic pair  $(g, \nabla_0) \in \mathcal{S} \times \Omega^1$  of metric and connection on  $L_C$ , the moduli space  $\mathcal{M}_1^{g,C}(E)$  is smooth outside  $\mathcal{M}_2^{g,C}(E)$  of expected codimension (2.4).

Moreover,  $\mathcal{M}^g(E)$  admits a natural orientation (see [D 1] and [K]). But  $\mathcal{M}_1^{g,C}(E)$  admits the special orientation because its normal bundle (2.4) has a natural complex structure. This orientation is described in details in section 5 of Ch.2 of [P-T].

Now, we need the usual restrictions on the topology of  $M$ . We will suppose

$$b_2^+(M) = 2p_g(M) + 1 \quad (2.5)$$

to be odd. Then both  $v.\dim \mathcal{M}^g(E) = 2d$  and  $v.\dim \mathcal{M}_1^{g,C}(E) = 2d_1$  must be even .

To compute the value of  $\gamma_1^{g,C}(E)$  evaluated at an argument  $(\sigma_1, \dots, \sigma_{d_1})$  we need to consider Donaldson's realisation of this collection of 2-cycles as a collection of smoothly embedded Riemannian surfaces  $(\Sigma_1, \dots, \Sigma_{d_1})$  which are in general position in the following sense :

1) Any two surfaces  $\Sigma_i$  and  $\Sigma_j$  meet transversally. And let

$$\{m_1, \dots, m_N\} \quad (2.6)$$

be the set of all points which are intersection points for some  $i$  and  $j$ .

2) exactly two surfaces pass through any point of intersection  $m_i$  so that in the flag diagram

$$\begin{array}{ccc} & \{m_i \in \Sigma_j\} & \\ & \swarrow \quad \searrow & \\ \{m_i, \dots, m_N\} & & \{\Sigma_1, \dots, \Sigma_{d_1}\} \end{array} \quad (2.7)$$

the projection to the set of all intersection points is an "unramified double cover";

For every 2-cocycle  $\sigma$  Donaldson constructed a so called fundamental cycle  $D_\sigma$  in the space  $\mathcal{B}(E)$  of gauge-orbits, which is a closed subspace of codimension 2 in  $\mathcal{B}(E)$  (see [D 1] or the formulas (4.18) - (4.20) in the survey article [T 1]). Then the third condition is

3) the collection of fundamental cycles  $D_{\sigma_1}, \dots, D_{\sigma_{d_1}}$  is in general position with respect to the strata of ends of  $\mathcal{M}_1^g(E)$ .

Then we can define the value of the  $Spin^{\mathfrak{C}}$ -polynomial as the algebraic number of points of intersection

$$\gamma_1^{g,C}(E)(\sigma_1, \dots, \sigma_{d_1}) = D_{\sigma_1} \cap \dots \cap D_{\sigma_{d_1}} \cap \mathcal{M}_1^{g,C}(E) \quad (2.8)$$

This definition makes sense because of

**Analogue of Donaldson's Lemma .** 1) If  $\Sigma_1, \dots, \Sigma_{d_1}$  are chosen in general position and

$$c_2(E) \geq \frac{3}{2}(b_2^+ + 1) - \frac{1}{2}c_1.C - 2\chi_C(L_0) \quad (2.9)$$

then the intersection (2.8) is compact.

2) If  $g_t$  is a general path in the space of metrics (which doesn't intersect walls (see (2.19))), then the union of all intersections (2.8) is smooth and compact.

3) The intersection number (2.8) depends only on the homology classes of the  $\sigma_i$ 's.

**Proof.** Since we use the same ideas and constructions as in [D 1] we will prove only first statement where our constants are a little bit different from Donaldson's. The proof of other statements is left to the reader.

**Remark.** In our applications we will use the  $SO(3)$  - bundles with  $w_2(E) \neq 0$ , so one doesn't even need the estimate (2.9).

If the intersection (2.8) is not compact, then there exists a sequence of connections

$$\{a_i\} \in \bigcap_{i=1}^{d_1} D_{\sigma_i} \cap \mathcal{M}_1^{g,C}(E)$$

which after suitable gauge transformations will converge uniformly (with bounded norm of the curvature) on  $M - \{m_1, \dots, m_l\}$ , where  $\{m_i\}$  is a finite set of points of  $M$ , which can be regularized by an  $L^p$  - gauge transformation. The limit connection can be regularized to an anti self dual connection

$$a_\infty \in \mathcal{M}_1^{g,C}(E') \text{ with}$$

$$c_1(E') = c_1(E), c_2(E') = c_2(E) - l$$

$$\chi_C(E') = \chi_C(E) + l$$

(because of the regularisation theorem of Uhlenbeck [ U ]).

Moreover  $l \leq c_2 - \frac{1}{4}c_1^2$  (otherwise  $\mathcal{M}^g(E')$  is empty by the Bogomolov inequality).

Since every  $D_{\sigma_i}$  is closed, there are two possibilities:

$$\text{either } a_\infty \in D_{\sigma_i} \quad (2.10)$$

$$\text{or } \exists j \text{ such that } m_j \in \Sigma_i$$

Consider first the extremal case  $l = c_2 - \frac{1}{4}c_1^2 + 1$ . Then the flag diagram (2.7) gives the inequality

$$2l = \#\{m_i \in \Sigma_j\} \geq \#\{\Sigma_1, \dots, \Sigma_{d_1}\} \quad (2.11)$$

But

$$d = \frac{1}{2} \dim_{\mathbb{R}} \mathcal{M}^g(E) = 4c_2 - c_1^2 - \frac{3}{2}(b_2^+ + 1) \quad (2.12)$$

$$d_1 = \frac{1}{2} \dim_{\mathbb{R}} \mathcal{M}_1^{g,C}(E) = 3c_2 - 1 - \frac{1}{2}c_1(c_1 - C) - \frac{3}{2}(b_2^+ + 1) - 2\chi_C(L_0)$$

From this the inequality (2.11) is equivalent to

$$c_2 < \frac{3}{2}(b_2^+) - \frac{1}{2}c_1.C - 2\chi_C(L_0) \quad (2.13)$$

contradicting (2.9).

In the general case

$$a_\infty \in \bigcap_{i=1}^{l'} D_{\sigma_i} \quad (2.14)$$

and each surface  $\Sigma_{l'+1}, \dots, \Sigma_{d_1}$  contains at least one point in  $\{m_1, \dots, m_l\}$ .

By the general position condition 3)

$$\bigcap_{i=1}^{l'} D_{\sigma_i} \cap \mathcal{M}_1^{g,C}(E') \neq \emptyset \Rightarrow \frac{1}{2} \dim_{\mathbb{R}} \mathcal{M}_1^{g,C}(E') \geq l' \quad (2.15)$$

On the other hand the flag diagram (2.7) gives the inequality

$$2l \geq d_1 - l' \iff l' \geq d_1 - 2l \quad (2.16)$$

From (2.15) and (2.16) we have

$$\frac{1}{2} \dim_{\mathbb{R}} \mathcal{M}_1^{g,C}(E') \geq \frac{1}{2} \dim_{\mathbb{R}} \mathcal{M}_1^{g,C}(E) - 2l$$

that is

$$4.c_2 - 4l - c_1^2 - 3(b_2^+ + 1) - 1 + \chi_C(E) + l \geq 4.c_2 - c_1^2 - 3(b_2^+ + 1) - 1 + \chi_C(E) - 2l$$

and this is a contradiction if  $l$  is positive.

This proves the lemma and completes - with the usual additions (see 3.1-3.3 from [D 1]) the construction of the Spin - polynomials

$$\gamma_1^{g,C}(2, c_1, c_2) \in S^{d_1} H^2(M, \mathbb{Z}) \quad (2.17)$$

for a regular (in the sense the Transversality Theorem ) metric  $g$  avoiding reducible connections.



Recall that if  $b_2^+ = 1$ , then associating to the metric  $g$  the ray of harmonic selfdual forms on  $M$  defines the so called period map of the space of Riemannian metrics to the Lobachevski space

$$K^+ \subset H^2(M, \mathbb{R})/\mathbb{R}^+ \quad (2.18)$$

The Lobachevski space  $K^+$  is divided by the collection of walls  $\{W_e = e^\perp\}$

$$e \in H^2(M, \mathbb{Z}), e \equiv c_1 \bmod 2, c_1^2 - 4c_2 \leq e^2 \leq 0 \quad (2.19)$$

into chambers of type  $(c_1, c_2)$ , which form a set  $\Delta$ .

Actually we can lift  $SO(3)$  - connections up to  $U(2)$ - connections, then the reducibility conditions give the decomposition of our vector bundle as  $E = L_e \oplus L_{c_1-e}$ . The wall  $e$  will be important for us if  $\chi_C(L_e)$  or  $\chi_C(L_{c_1-e})$  will be positive. Now we can compute the link of the singularity of the bordism of the moduli spaces given by one dimensional path in the parameter space  $\mathcal{S} \times \Omega^1$  and the number of points of the intersection (2.8) which disappeared (appeared) in (from) this singularity. This number is given by the pure topological formula actually by Porteus formula for the virtual index vector bundle of the family of Dirac operators (the details you can see in the forthcoming preprint of Victor Pidstrigach). From this by the same bordism arguments as in [D 1] and [K] we obtain the description of the dependence of the Spin-polynomials on the parameters:

**Theorem 2.1.** If  $b_2^+ = 1$  then for every pair of regular metrics  $g_1, g_2$  from the same chamber  $C \in \Delta$

$$\gamma_1^{g_1, C}(2, c_1, c_2) = \gamma_1^{g_2, C}(2, c_1, c_2) \quad (2.20)$$

(if  $b_2^+ \geq 3$  this is true without any chamber condition).

On the other hand the dependence of the Spin-polynomials on changing the  $Spin^{\mathbb{C}}$ -structure  $C \in H_w(M)$  was given in section 1:

$$\gamma_1^{g_1, C+2\delta}(2, c_1, c_2) = \gamma_1^{g_1, C}(2, c_1 - 2\delta, c_2 - c_1 \cdot \delta + \delta^2) \quad (2.21)$$

by the formula (1.12).

### 3 Algebraic surfaces

If  $M$  is the underlying manifold of an algebraic surface  $S$ , then there exist the canonical  $Spin^{\mathbb{C}}$ -structure given by the anticanonical class  $-K_S$  (we will drop the index as long as there is no danger of confusion).

In this case for Hodge metric  $g_H$  given by a polarization

$$H \in PicS \subset H^2(S, \mathbb{Z}) \quad (3.1)$$

the Donaldson-Uhlenbeck identification theorem gives

$$\mathcal{M}^{g_H}(E) = M^H(2, c_1, c_2) \quad (3.2)$$

where the right side is the moduli space of  $H$  - slope stable bundles on  $S$  with Chern classes  $c_1, c_2$ .

Under the identification (3.2)  $(a) = E$  we have an identification

$$ker D_a^{g_H, -K} = H^0(E) \oplus H^2(E) \quad (3.3)$$

$$coker D_a^{g_H, -K} = H^1(E)$$

where the  $H^i(E)$  denote the coherent cohomology groups and

$$ind D_a^{g_H, -K} = \chi(E) \quad (3.4)$$

$$\chi_{-K}(L_0) = \chi(\mathcal{O}_S) = p_g + 1$$

The subspace (2.2) then is the Brill - Noether locus

$$\mathcal{M}_1^{g_H, -K}(2, c_1, c_2) = \{E \in M^H(2, c_1, c_2) | h^1(E) \geq -\chi(E) + 1\} \quad (3.5)$$

But in the situation of surfaces the last inequality

$$h^1(E) \geq -\chi(E) + 1 \iff h^0(E) + h^2(E) \geq 1 \quad (3.6)$$

Hence we have a decomposition

$$\mathcal{M}_1^{g_H, -K}(2, c_1, c_2) = M_{1,0}^H(2, c_1, c_2) \cup M_{0,1}^H(2, c_1, c_2)$$

where the components are algebraic subvarieties

$$M_{1,0}^H(2, c_1, c_2) = \{E \in M^H(2, c_1, c_2) | h^0(E) \geq 1\} \quad (3.7)$$

$$M_{0,1}^H(2, c_1, c_2) = \{E \in M^H(2, c_1, c_2) | h^2(E) \geq 1\}$$

On the other hand the transformation

$$E \rightsquigarrow E^*(K) = E^* \otimes \mathcal{O}_S(K) \quad (3.8)$$

gives the identification

$$M^H(2, c_1, c_2) = M^H(2, 2K - c_1, c_2 - c_1.K + K^2) \quad (3.9)$$

and by Serre-duality

$$M_{0,1}^H(2, c_1, c_2) = M_{1,0}^H(2, 2K - c_1, c_2 - c_1.K + K^2) \quad (3.10)$$

By this reason we have in the algebraic geometric situation two polynomials

$$\gamma_{1,0}^{g_H, -K}(2, c_1, c_2), \text{ and } \gamma_{0,1}^{g_H, -K}(2, c_1, c_2) \quad (3.11)$$

given by the construction in section 2 with the subspaces (3.7) ( of course, if our Hodge metric  $g_H$  is regular, then the spaces (3.7) have the expected dimension ).

Now to compute the Spin-polynomial (2.17) we must sum the individual polynomials (3.11) but here we must be careful because the natural orientations of the components (3.7) can be different .Actually in section 5 of Ch.1 of [P-T] following the orientation law was proved:

**Orientation Rules** .1) If the number

$$1 - \chi(E) \quad (3.12)$$

is even, then the natural orientations of  $M_{1,0}^H(2, c_1, c_2)$  and  $M_{0,1}^H(2, c_1, c_2)$  coincide (compared with the complex orientation).

2) otherwise they have different orientations.

It means that

$$c_2 = \frac{1}{2}(c_1^2 - c_1.K) + 1 \text{ mod } 2 \Rightarrow \gamma_1^{g_H, -K} = \gamma_{1,0}^{g_H, -K} + \gamma_{0,1}^{g_H, -K} \quad (3.13)$$

and

$$c_2 = \frac{1}{2}(c_1^2 - c_1.K) + \text{mod}2 \Rightarrow \gamma_1^{g_H, -K} = \gamma_{1,0}^{g_H, -K} - \gamma_{0,1}^{g_H, -K}$$

On the analogy of the Non-degeneracy Theorem for the original Donaldson polynomial we can prove

**Theorem 3.1.** Assume our Hodge metric  $g_H$  avoids reducible connections and

$$1) c_2 \geq 5(p_g + 1) + \frac{1}{2}c_1.K \quad (\text{see}(2.9);$$

$$2) c_2 = \frac{1}{2}(c_1^2 - c_1.K) + 1 \text{mod}2$$

$$3) M_{1,0}^H(2, c_1, c_2) \quad \text{and} \quad M_{0,1}^H(2, c_1, c_2)$$

have an expected positive dimension, then

$$\gamma_1^{g_H, -K}(2, c_1, c_2) \neq 0$$

**Proof.** Because of condition 1) the polynomial exists. We can choose some smooth curve  $C$  in the complete linear system  $|NH|$ ,  $N \gg 0$ , such that the restriction map

$$\text{res}_C : M_{1,0}^H(2, c_1, c_2) \cup M_{0,1}^H(2, c_1, c_2) \rightarrow M_C(2, c_1.C) \quad (3.14)$$

is an embedding (see, for example, [T 3]). On the other hand, there is an ample divisor  $\Theta \in \text{Pic}M_C(2, c_1.C)$  and the value of Spin-polynomial on the class  $C$  is the sum of degrees of the image subvarieties  $(\text{res}_C(M_{1,0}^H(2, c_1, c_2) \cup M_{0,1}^H(2, c_1, c_2)))$  with respect to this  $\Theta$ .

It must be a sum, not a difference because of condition 2) (see (3.13)). As in the case of the original Donaldson polynomials we are done.

The condition 3) is very important because the properties of Spin-polynomials depend on degrees

$$\deg_H c_1 = H.c_1, \quad \deg_H K_S = H.K_S \quad (3.15)$$

with respect to the polarisation  $H$ . Namely, in contrast to the behaviour of the Donaldson polynomials the Spin-polynomials may vanish for all values of  $c_2$ :

**Lemma 3.1** If

$$2K_S.H \leq c_1.H \leq 0 \quad (3.16)$$

then

$$M_{1,0}^H(2, c_1, c_2) \cup M_{0,1}^H(2, c_1, c_2) = \emptyset$$

and ,hence

$$\gamma_1^{g_H, -K}(2, c_1, c_2) = 0$$

for every  $c_2$ .

**Proof.** Indeed,

$$h^0(E) > 0 \iff \exists s : \mathcal{O}_S \rightarrow (E), s \neq 0$$

On the other hand

$$h^2(E) > 0 \iff \exists j : E \rightarrow \mathcal{O}_S(K_S), j \neq 0$$

but  $s \neq 0, c_1.H \leq 0$  contradicts the stability condition for  $E$  and  $c_1.H \geq 2H.K_S$  is a contradiction to the stability condition for  $E$ , too. We are done.

**Remark.** Of course, the inequalities (3.16) are possible for rational surfaces only. The vanishing condition (3.16) is crucial (it is actually due to Donaldson [D 2]). Of course the original Donaldson polynomials don't vanish under this conditions as for example in the case  $S = \mathbb{CP}^2$  for the sequence

$$(2, -2, c_2), c_2 \in \mathbb{Z}^+$$

## 4 Asymptotic regularity

Let  $S$  be a algebraic surface,  $H$  a polarisation on  $S$  and  $c_1 \in \text{Pic}S$  a divisor class.

**Definition 4.1** A class  $c_1 \in \text{Pic}S$  is called  $H$ -semisimple, if for any effective curve  $C \subset S$

$$c_1.H > 2C.H \iff C.K_S + C^2 \leq c_1.C \quad (4.1)$$

(I would like to emphasize that the left side of the last inequality is the degree of the canonical class on  $C$ ).

On the analogy of Donaldson's Non-degeneracy Theorem we prove

**Theorem 4.1.** For every H-semisimple  $c_1 \in PicS$  with  $c_1.H > 0$  there exists a constant  $N(H, c_1)$  such that for  $c_2 \geq N(H, c_1)$

$$v.dim M_{1,0}^H(2, c_1, c_2) = dim M_{1,0}^H(2, c_1, c_2) > 0 \quad (4.2)$$

and general point of  $M_{1,0}^H(2, c_1, c_2)$  is smooth.

**Proof 7** Each  $E \in M_{1,0}^H(2, c_1, c_2)$  has a section, that is a non zero homomorphism

$$s : \mathcal{O}_S \rightarrow (E) \quad (4.3)$$

The subscheme of zeroes of this homomorphism contains a priori subschemes of different dimensions:

$$(s)_0 = C \cup \xi \text{ with } dim C = 1, dim \xi = 0 \quad (4.4)$$

Because  $E$  is H-stable, we have

$$2C.H < c_1.H \quad (4.5)$$

There exists a finite set of non empty complete linear systems

$$|0|, |C_1|, \dots, |C_N| \quad (4.6)$$

satisfying the inequality (4.5) (  $|0|$  is the complete linear system of the class  $0 \in PicS$  ).

For every  $i = 0, 1, \dots, N$  consider the variety

$$GAM_{C_i}(2, c_1, c_2) = \{0 \rightarrow \mathcal{O}_S(C_i) \rightarrow E \rightarrow J_\xi(c_1 - C_i) \rightarrow 0\} / \mathbb{C}^* \quad (4.7)$$

of all non trivial extensions up to homotheties, where  $J_\xi$  is the ideal sheaf of a 0-dimensional subscheme  $\xi$  (of a cluster  $\xi$  for short). (GAM alias GAM-BURGER ).

We need to prove that

$$\dim \bigcup_{i=0}^N GAM_{C_i}(2, c_1, c_2) \leq v.\dim M_{1,0}^H(2, c_1, c_2) \quad (4.8)$$

and that  $M_{1,0}^H(2, c_1, c_2) \neq \emptyset$ .

But under the operation  $E \rightsquigarrow E(-C_i)$

$$GAM_{C_i}(2, c_1, c_2) = GAM_0(2, c_1 - 2C_i, c_2 - c_1.C_i + C_i^2) \quad (4.9)$$

The constants  $\{C_i^2 - c_1.C_i\}$  are bounded, hence we are done if we prove the following

**Lemma 4.1** For  $c_2 \gg 0$

$$\begin{aligned} \dim GAM_0(2, c_1, c_2) &\leq v.\dim M_{1,0}^H(2, c_1, c_2) = \\ &= 3c_2 - 1 - \frac{c_1(c_1 + K)}{2} - (p_g + 1) \end{aligned} \quad (4.10)$$

(see (2.12) with  $C = -K_S$ )

Note that

$$\begin{aligned} 3(c_2 - c_1.C_i + C_i^2) - 1 - \frac{(c_1 - 2C_i)(K + c_1 - 2C_i)}{2} - (p_g + 1) &= \\ = 3c_2 - 1 - \frac{c_1(c_1 + K)}{2} - (p_g + 1) + (C_i.K + C_i^2 - c_1.C_i) \end{aligned}$$

and the tail is non positive due to the inequality (4.1).

**Proof of Lemma 4.1** The natural projection

$$\pi : GAM_0(2, c_1, c_2) \rightarrow \text{Hilb}^{c_2} S \quad (4.11)$$

given by sending the extension (4.7) to the cluster  $\xi$  as element of the Hilbert scheme is surjective for big  $c_2$ . A fibre

$$\pi^{-1}(\xi) = \mathbb{P}Ext^1(J_\xi(c_1), \mathcal{O}_S) = \mathbb{P}H^1(J_\xi(c_1 + K))^* \quad (4.12)$$

by Serre-duality.

For every  $\xi \in Hilb^{c_2}S$  we have a short exact sequence

$$0 \rightarrow J_\xi(c_1 + K) \rightarrow \mathcal{O}_S(c_1 + K) \rightarrow \mathcal{O}_\xi(c_1 + K) \rightarrow 0$$

giving rise to a cohomology exact sequence

$$H^0(J_\xi(c_1 + K)) \rightarrow H^0(\mathcal{O}_S(c_1 + K)) \rightarrow \mathbb{C}^{c_2} \rightarrow H^1(J_\xi(c_1 + K)) \rightarrow H^1(\mathcal{O}_S(c_1 + K)) \rightarrow 0 \quad (4.13)$$

Moreover,

$$h^0(J_\xi(c_1 + K)) = 0 \Rightarrow \dim \mathbb{P}H^1(J_\xi(c_1 + K)) = c_2 - \chi(\mathcal{O}_S(c_1 + K)) - 1 =$$

$$= c_2 - 1 - \frac{c_1(c_1 + K)}{2} - (p_g + 1) \quad (4.14)$$

Consider the subvariety

$$\Delta = \{\xi \in Hilb^{c_2}S \mid h^0(J_\xi(c_1 + K)) > 0\} \quad (4.15)$$

It is easy to see that

$$\dim \Delta \leq c_2 + \dim |c_1 + K| = c_2 + h^0(\mathcal{O}_S(c_1 + K)) - 1$$

On the other hand from (4.13) we have

$$\dim \pi^{-1}(\xi) = \mathbb{P}H^1(J_\xi(c_1 + K)) \leq h^1(\mathcal{O}_S(c_1 + K)) + c_2$$

Hence

$$\dim \pi^{-1}(\Delta) \leq 2c_2 - 2 + h^0(\mathcal{O}_S(c_1 + K)) + h^1(\mathcal{O}_S(c_1 + K))$$

and



$$c_2 > 2h^0(O_S(c_1 + K)) \Rightarrow \dim \pi^{-1}(\Delta) < 3c_2 - 2 - \frac{c_1(c_1 + K)}{2} - (p_g + 1).$$

This proves Lemma 4.1. To finish the proof of Theorem 4.1 we prove

**Lemma 4.2** .If  $c_1.H > 0$ , then for  $c_2 \gg 0$

$$M_{1,0}^H(2, c_1, c_2) \neq \emptyset$$

and hence by Theorem 4.1 it has the expected dimension.

**Proof** .We need to prove that for generic  $\xi \in \text{Hilb}^{c_2}$  and  $c_2 \gg 0$  any non trivial extension

$$0 \rightarrow \mathcal{O}_S \rightarrow E \rightarrow J_\xi(c_1) \rightarrow 0$$

is H-stable. Twisting E by  $(-c_1)$  we have

$$0 \rightarrow \mathcal{O}_S(-c_1) \rightarrow E(-c_1) \rightarrow J_\xi \rightarrow 0$$

The hypothetical destabilizing line bundle must be of type  $\mathcal{O}_S(-C)$ , where C is an effective curve subject to the inequality (4.1) and the cluster  $\xi$  must be supported on this effective curve.

But the collection (4.6) of such curves is finite and for  $c_2 \gg 0$  (as in Theorem 4.1) a generic  $\xi$  is not contained in any curve in this collection of complete linear systems (see (4.15)).

**Definition 4.2** . A class  $c_1 \in \text{Pic}S$  is called H-simple, if it is semisimple and the class  $2K_S - c_1$  is semisimple too.

As a corollary of Theorems 3.1 and 4.1 we provide

**Theorem 4.2**. Assume that our Hodge metric g avoids reducible connections . Then for every H-simple  $c_1 \in \text{Pic}S$  with  $c_1.H > 0$  there exists a constant  $N(H, c_1)$  such that for

$$c_2 \geq N(H, c_1), c_2 = \frac{1}{2}(c_1^2 - c_1.K) + p_g \text{ mod } 2 \quad (4.16)$$

$$\gamma_1^{g_H, -K}(2, c_1, c_2) \neq 0$$

At last (but not at least ) we need to explain what we have to do if our Hodge metric does not avoid reducible connections. Certainly in case when  $rk \text{Pic}S > 1$  we may use the following extremely useful trick:

**Definition 4.3.** A polarization  $H^\varepsilon$  is called close to  $H$  if the ray  $\mathbb{R}^+.H^\varepsilon$  in  $K^+$  (2.18) is close to the ray  $\mathbb{R}^+.H$  in Lobachevski metric.

**Lemma 4.3.** If a class  $c_1 \in \text{Pic}S$  is a  $H$ -simple, then for a polarisation  $H^\varepsilon$  sufficiently close to  $H$

1)  $c_1$  is  $H^\varepsilon$ -simple too.

$$2) \ 2K_S.H < c_1.H < 0 \implies 2K_S.H^\varepsilon < c_1.H^\varepsilon < 0 \quad (4.17)$$

3) for every polarisation  $H$  there exists a sufficiently close to  $H$  polarisation  $H^\varepsilon$  such that the Hodge metric  $g_{H^\varepsilon}$  avoids the reducible connections.

**Proof.** For a sufficiently close polarisation  $H^\varepsilon$

$$2C.H < c_1.H \implies 2C.H^\varepsilon < c_1.H^\varepsilon \quad (4.18)$$

Hence the collection of linear systems (4.6) for  $H^\varepsilon$  is the same as for  $H$  and we have 1) and 2).

To prove 3) it is enough to remark that the set of rays of polarisations is dense in the projectivisation of the Kähler cone in  $K^+$  and the set of walls is discrete and locally finite.

In the last section we consider three very simple examples to show how we can use Theorem 4.2 and Lemma 3.1 to distinguish the underlying smooth structures of rational surfaces and surfaces of general type.

## 5 Applications

For the beginning we prove

**Theorem 5.1.** If an algebraic surface  $S$  is diffeomorphic to  $\mathbb{CP}^2$  then  $S = \mathbb{CP}^2$  (as algebraic surface).

**Proof.** Let

$$f : \mathbb{CP}^2 \rightarrow S \quad (5.1)$$

be a diffeomorphism,  $h \in \text{Pic}S$  be the positive generator of  $\text{Pic}S$  ( $h^2 = 1$ ). Then (using the real anti involution of  $\mathbb{CP}^2$  if necessary) we may consider the case when

$$f^* (h) = l$$

is the class of the line on  $\mathbb{CP}^2$ .

For the canonical class we have  $K_S = 3h$  (otherwise  $K_S = -3h$  and  $S$  is rational).

Then the increment of the canonical class with respect to  $f$  is

$$\delta_f(K) = \frac{f^*(K_S) - K_{\mathbb{CP}^2}}{2} = -K_{\mathbb{CP}^2}^2 \quad (5.2)$$

Consider the following topological type of vector bundles on  $S$

$$(2, h, c_2), c_2 \gg 0 \quad (5.3)$$

**Lemma 5.1** .For all  $c_2$  the Spin-polynomial

$$\gamma_1^{g_h, -K}(2, h, c_2) = 0.$$

**Proof.**The operation  $f^*$  gives the equality

$$\gamma_1^{g_h, -K}(2, h, c_2) = \gamma_1^{f^*(g_h), K_{\mathbb{CP}^2}}(2, l, c_2)$$

By the equality (2.20)

$$\gamma_1^{f^*(g_h), K_{\mathbb{CP}^2}}(2, l, c_2) = \gamma_1^{g_{F-S}, K_{\mathbb{CP}^2}}(2, l, c_2)$$

where  $g_{F-S}$  is the Fubini-Study metric on  $\mathbb{CP}^2$ .

By the equality (2.21)

$$\gamma_1^{g_{F-S}, K_{\mathbb{CP}^2}}(2, l, c_2) = \gamma_1^{g_{F-S}, -K_{\mathbb{CP}^2}}(2, 2K_{\mathbb{CP}^2} + l, c_2 + 6) = \gamma_1^{g_{F-S}, -K_{\mathbb{CP}^2}}(2, -5l, c_2 + 6)$$

The first Chern class  $c_1 = -5l$  satisfies the inequality (3.16) and by Lemma 3.1 we are done.

To provide a contradiction to the existence of an  $f$  (5.1) we prove

**Lemma 5.1'** .On  $S$  the class  $h$  is  $h$ -simple. Hence by Theorem 4.2 if  $c_2$  is odd then

$$\gamma_1^{g_h, -K}(2, h, c_2) \neq 0$$

**Proof** For  $h$  the set of linear systems (4.6) is  $|0|$ . Hence  $h$  is  $h$ -semisimple. For the class  $2K_S - h = 5h$  the set (4.6) is

$$|0|, |h|, |2h|$$

For these classes we have

$$h^2 + h.K_S = 4 < 5, (2h)^2 + 2h.K_S = 10 \leq 10$$

and thus  $2K_S - h$  is semisimple too. This implies that  $h$  is simple. We are done by Theorem 4.2.

Let  $F_1$  be the projective plane blown up in one point (Hirzebruch surface of number 1).

**Theorem 5.2.** If an algebraic surface  $S$  is diffeomorphic to  $F_1$  then  $S = F_{2n+1}$  that is the odd Hirzebruch surface.

**Proof.** Let

$$f : F_1 \rightarrow S \quad (5.4)$$

be a diffeomorphism. We can find a basis  $h, e$  in  $PicS$  such that

$$h^2 = 1, e^2 = -1, h.e = 0, K_S = 3h - e \quad (5.5)$$

Again it is sufficient to consider the case when

$$f^*(h) = l, f^*(e) = E \quad (5.6)$$

where  $l$  is the class of line and  $E$  is the exceptional divisor on  $F_1$ . Hence

$$K_{F_1} = -3l + E = f^*(-K_S) \quad (5.7)$$

Then the increment of the canonical class

$$\delta_f K = -K_{F_1}$$

We only need to investigate the case when  $S$  is a surface of the general type and minimal.

Then  $K_S$  is a polarisation on  $S$  and  $f^*(K_S) = -K_{F_1}$  is a polarisation on  $F_1$ . Let  $H$  be a polarisation on  $S$  sufficiently close to  $K_S$  such that

$$f^*(H) = H_1$$

is a polarisation on  $F_1$  sufficiently close to  $(-K_{F_1})$ .

**Lemma 5.2.** For all  $c_2$  the Spin-polynomial

$$\gamma_1^{g_H, -K_S}(2, h, c_2) = 0.$$

**Proof.** The operation  $f^*$  gives the equality

$$\gamma_1^{g_H, -K_S}(2, h, c_2) = \gamma^{f^*(g_H), K_{F_1}}(2, l, c_2).$$

But the metrics  $f^*(g_H)$  and  $g_{H_1}$  on  $F_1$  are contained in the same chamber. More precisely they have the same image of the period map. Then by (2.20)

$$\gamma^{f^*(g_H), K_{F_1}}(2, l, c_2) = \gamma^{g_{H_1}, K_{F_1}}(2, l, c_2)$$

Moreover

$$\gamma^{g_{H_1}, K_{F_1}}(2, l, c_2) = \gamma^{g_{H_1}, -K_{F_1}}(2, -5h + 2l, c_2 + 5)$$

The first Chern class  $c_1 = -5h + 2l$  satisfies the inequality (3.16). Thus by Lemma 3.1 we are done.

To provide a contradiction to existence of  $f$  (5.4) we prove

**Lemma 5.2'.** On  $S$  the class  $h$  is  $K_S$ -simple.

**Proof** Let  $C$  be an effective curve  $C$  of the form  $C = xh - ye$ . We will check whether the inequality (4.1) does hold for  $C$ : First

$$2 \leq 2C.K_S \leq h.K_S = 3 \Rightarrow C.K_S = 1 \Rightarrow y = 3x - 1$$

Then

$$C^2 = x^2 - y^2 = -8x^2 + 6x - 1, C^2 + C.K_S = -8x^2 + 6x < C.h = x$$

for every  $x \in \mathbb{Z}$ . Hence  $h$  is  $K_S$ -semisimple.

Now for  $c_1 = 2K_S - h = 5h - 2l$  we have

$$2 \leq 2C.K_S \leq 13 \Rightarrow C.K \in \{1, \dots, 6\} \Rightarrow y = 3x - \{1, \dots, 6\}$$

Then

$$C^2 + C.K_S = -8x^2 + 6x.\{1, \dots, 6\} - \{0, 2, 6, 12, 20, 30\}$$

$$C.(5h - 2e) = -x + 2.\{1, \dots, 6\}$$

It is easy to check that for any of the six cases the inequality (4.1) holds. Hence  $5h - 2e$  is  $K_S$ -semisimple, too, and we are done.

At last let us go to the Hirzebruch Problem.

**Theorem 5.3.** On  $S^2 \times S^2$  there exists the unique algebraic structure  $Q = \mathbb{CP}^1 \times \mathbb{CP}^1$  up to the elementary transformations to the even Hirzebruch surface  $F_{2n}$ .

**Proof.** In this case for any topological type  $(2, c_1, c_2)$  of vector bundle the virtual dimension of  $M^H(2, c_1, c_2)$  is odd. We will use a simple trick:

Let

$$f : Q \rightarrow S \quad (5.8)$$

be a diffeomorphism. We can find basis  $h_+, h_-$  in  $PicS$  such that

$$h_+^2 = h_-^2 = 0, h_+.h_- = 1, K_S = 2h_+ + 2h_- \quad (5.9)$$

such that  $f^*(h_+) = h'_+$ ,  $f^*(h_-) = h'_-$  is the standard basis of  $Pic\mathbb{CP}^1 \times \mathbb{CP}^1$  and  $f^*(K_S) = -K_Q$ .

We only need to consider the case when  $K_S$  is nef.

Let's blow up a point  $p$  on  $Q$  and  $f(p)$  on  $S$ . Then the diffeomorphism  $f$  (5.8) can be extended to a diffeomorphism

$$\tilde{f} : \tilde{Q} \rightarrow \tilde{S} \quad (5.8')$$

and

$$K_{\tilde{Q}} = K_Q + E', K_{\tilde{S}} = K_S + E \quad (5.10)$$

where  $E$  and  $E'$  are the respective exceptional curves. The increment of the canonical class with respect to  $\tilde{f}$  is

$$\delta_{\tilde{f}}K = -K_Q$$

The divisor class  $(h_+ + h_-) \in \text{Pic}\tilde{S}$  is nef and we consider a polarisation  $H$  on  $\tilde{S}$  sufficiently close to  $(h_+ + h_-)$  such that  $\tilde{f}^*(H) = H'$  is a polarisation on  $\tilde{Q}$  sufficiently close to  $(h'_+ + h'_-)$ .

**Lemma 5.3.** For all  $c_2$  the Spin-polynomial

$$\gamma_1^{g_H, -K_S - E}(2, h_+ + h_- + E, c_2) = 0$$

**Proof** As usual by (2.20) and (2.21)

$$\gamma_1^{g_H, -K_S - E}(2, h_+ + h_- + E, c_2) = \gamma_1^{g'_H, -K_Q - E'}(2, -3(h'_+ + h'_- - E'), c_2 + 2)$$

The first Chern class  $c_1 = -3(h'_+ + h'_- - E')$  satisfies the inequality (3.16) so by Lemma 3.1 we are done.

To provide a contradiction to the existence of  $\tilde{f}$  (and hence of  $f$  (5.8)) as before we prove

**Lemma 5.3'.** On  $\tilde{S}$  the divisor class  $h_+ + h_- + E$  is  $H$  - simple.

**Proof.** For an effective curve  $C$

$$0 \leq C(h_+ + h_-) < 1 \Rightarrow C = m(h_+ - h_-) + nE$$

But then

$$C.K_{\tilde{S}} + C^2 = -2m^2 - n^2 - n \leq C.(h_+ + h_- + E) = -n$$

and  $h_+ + h_- + E$  is  $H$  - semisimple.

Now for  $2K_{\tilde{S}} - c_1 = 3(h_+ + h_-) + E$  let

$$C = xh_+ + m(h_+ - h_-) + nE.$$

Then

$$0 \leq C(h_+ + h_-) < 3 \Rightarrow x = \{0, 1, 2\}$$

and

$$C.K_{\tilde{S}} + C^2 = \{0, 2, 4\}m - 2m^2 - n^2 + \{0, 2, 4\} - n, \quad C.(3h_+ + 3h_- + E) = \{0, 3, 6\} - n$$

It is easy to check that the right side of the inequality (4.1) holds for all  $n$  and  $m$ . Hence  $h_+ + h_- + E$  is  $H$ -simple. The reader may continue these purely arithmetical investigations himself. Good luck!.

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